METU - NCC

DIFFERENTIAL EQUATIONS MIDTERM 2	
Code : *MAT 219* Acad.Year: *2014-2015* Semester : *FALL* Date : *07.12.2014* Time : *9:40* Duration : *120 min*	Last Name: List #: Name : Student # : Signature : 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1. (16)	2. (18)	3. (14)	4. (18)	5. (18)	6. (16)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1A. *(4pts)* Convert the following system of differential equations to a matrix differential equation of the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$:

$$x_1' = \frac{x_2}{1+t^2} + e^t x_1 + \sin(t), \qquad \text{with} \qquad x_1(0) = 2$$
$$x_2' = x_1 + \arctan(t) \qquad\qquad x_2(0) = 4$$

$$x' = \begin{bmatrix} e^t & \frac{1}{1+t^2} \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \sin t \\ \arctan t \end{bmatrix} \quad \text{with} \quad x(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

1B. *(6+6pts)* Write the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is
(i) 3×3, with eigenvalues $\lambda = 2, 2, -5$ whose eigenvectors are $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -7 \end{bmatrix}$.

$$x = c_1 \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ -9 \\ 4 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 6 \\ -7 \end{bmatrix} e^{-5t}$$

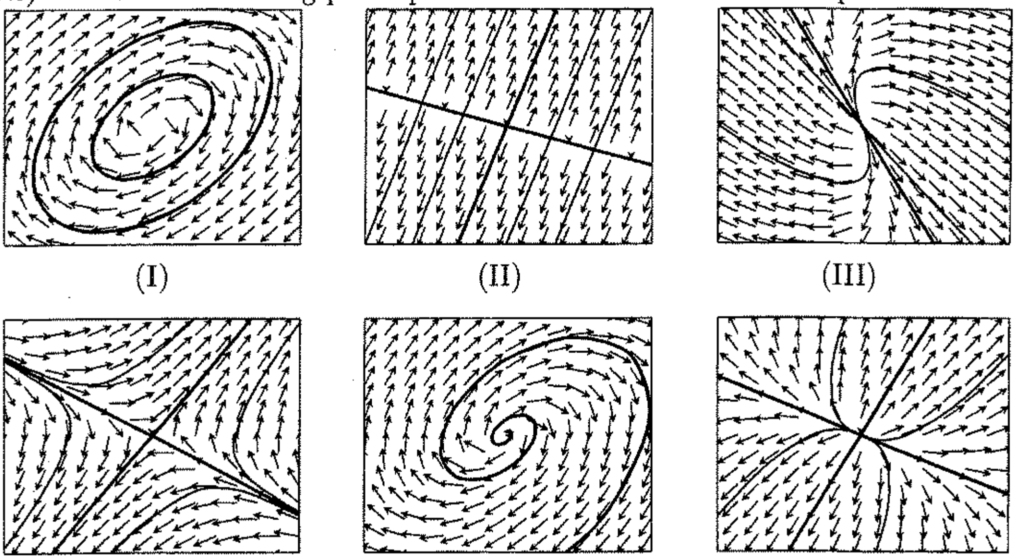
(ii) 5×5, with $\lambda = i, -i, i, -i, 0$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 2+i \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 2-i \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1984 \end{bmatrix}$.

$$-i(\cos t + i\sin t) = \sin t + i(-\cos t)$$

$$(2+i)(\cos t + i \sin t) = (2\cos t - \sin t) + i(\cos t + 2\sin t)$$

$$x = c_1 \begin{bmatrix} \cos t \\ \sin t \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 7\cos t \\ 0 \\ 2\cos t - \sin t \\ 3\cos t \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 7\sin t \\ 0 \\ \cos t - 2\sin t \\ 3\sin t \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1984 \end{bmatrix}$$

2A. *(6+2pts)* Match the following phase portraits with their differential equations.



- $\boxed{\text{VI}}$ $\quad \mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}\mathbf{x}$

  $\lambda^2 - 11\lambda + 24 = 0$
  $(\lambda - 3)(\lambda - 8) = 0$

- $\boxed{\text{IV}}$ $\quad \mathbf{x}' = \begin{bmatrix} 1 & 6 \\ 6 & 6 \end{bmatrix}\mathbf{x}$

  $\lambda^2 - 7\lambda - 30 = 0$
  $-3, 10$

- $\boxed{\text{V}}$ $\quad \mathbf{x}' = \begin{bmatrix} -1 & 4 \\ -5 & 3 \end{bmatrix}\mathbf{x}$

  $\lambda^2 + 2\lambda + 17 = 0$
  $\lambda = +1 \pm 4i$

- $\boxed{\text{II}}$ $\quad \mathbf{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\mathbf{x}$

  $\lambda^2 - 10\lambda = 0 \quad \lambda = 0, 10$

- $\boxed{\text{III}}$ $\quad \mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 1 \end{bmatrix}\mathbf{x}$

  $\lambda^2 - 6\lambda + 9 = 0 \quad \lambda = 3, 3$

- $\boxed{\text{I}}$ $\quad \mathbf{x}' = \begin{bmatrix} -2 & 4 \\ -5 & 2 \end{bmatrix}\mathbf{x}$

  $\lambda^2 + 16 = 0 \quad \lambda = \pm 4i$

2B. *(10pts)* Find the general solution to $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$. To find a particular solution, use variation of parameters. Eigenvalues: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Eigenvectors: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A - iI = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ Eigenvector for i $= \begin{bmatrix} 1 \\ i \end{bmatrix}$

Eigenvector for $-i$ $= \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Fundamental Solutions = Real and Imaginary Parts of $\begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} = \begin{bmatrix} 1 \\ i \end{bmatrix}(\cos t + i \sin t)$

$= \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{x^{(1)}} + i \underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{x^{(2)}}$ = Fund. solns.

Particular solution (via variation of params)

$x_p = \psi u \qquad \psi = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, $\det \psi = 1$,

$\Rightarrow u' = \psi^{-1}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow u = \begin{bmatrix} 0 \\ t \end{bmatrix}$

$\underset{(\text{part})}{x} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} t\sin t \\ t\cos t \end{bmatrix} \Rightarrow x = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} t\sin t \\ t\cos t \end{bmatrix}$

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3 *(14pts)* Find the general solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix} \mathbf{x}$.

(The eigenvalues of the matrix above are $\lambda = 2, 2, 2$.)

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}}_{A} - 2I = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix}}_{A-2I} \quad \text{where } A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{(A-2I)v=0} \rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{(A-2I)v=0} \Rightarrow \begin{array}{l} v_1 = 0 \\ v_2 = 0 \\ v_3 = \text{free}. \end{array}$$

Eigenvector for 2: $\begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_3$.

$\Rightarrow$ NOT ENOUGH EIGENVECTORS!!

FUND. SOLNS.

- $\boxed{x^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}}$

- $(A-2I)(w) = v$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = 0 \\ 3v_1 + v_2 = 1 \Rightarrow v_2 = 1 \\ w_3 = \text{free}. \end{array}$$

$w = \begin{bmatrix} 0 \\ 1 \\ w_3 \end{bmatrix}$ choose $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- $\boxed{x^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t}}$

- $(A-2I)z = w$ choose $z = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} z_1 = 1 \\ 3z_1 + z_2 = 0 \end{array}$$

$\Rightarrow z_2 = -3$

$z_3 = \text{free}.$

$$\boxed{x^{(3)} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t}}$$

GENERAL SOLUTION:

$$X = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{t} \right\} + c_3 \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} \right\}$$

4. *(6+6+6pts)* This problem has three unrelated parts.

(A) Write the general solutions to the following homogeneous differential equations.

(i) $y'' - 2y' - 8y = 0.$

$r^2 - 2r - 8 = 0$ general solution

$(r-4)(r+2) = 0.$ $y = c_1 e^{4t} + c_2 e^{-2t}$

$r = 4$ $r = -2$

(ii) $y^{(4)} - y = 0$

$r^4 - 1 = 0$ $y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

$(r^2 - 1)(r^2 + 1) = 0$

$(r-1)(r+1)(r-i)(r+i) = 0$

(iii) $4y'' - 4y' + y = 0$

$4r^2 - 4r + 1 = 0$ $y = c_1 e^{t/2} + c_2 t e^{t/2}$

$(2r-1)^2 = 0.$

$r = \frac{1}{2}$

(B) Write the general solution to the 10^{th} order homogeneous differential equation

$$y^{(10)} - 4y^{(9)} + 4y^{(8)} - 20y^{(7)} + 50y^{(6)} - 28y^{(5)} + 92y^{(4)} - 12y^{(3)} + 45y^{(2)} = 0$$

(Hint: The characteristic polynomial factors as $r^2(r^2 + 2r + 5)(r - i)^2(r + i)^2(r - 3)^2$.)

$0 = r^2 + 2r + 5 = r^2 + 2r + 1 + 4 = (r+1)^2 + 4 \Rightarrow r = -1 \pm 2i$

$y = c_1 + c_2 t + c_3 e^{-t} \cos 2t + c_4 e^{-t} \sin 2t + c_5 \cos t + c_6 \sin t + c_7 t \cos t + c_8 t \sin t$
$+ c_9 e^{3t} + c_{10} t e^{3t}$

(C) Write a constant coefficient, linear **homogeneous** differential equation of lowest order which is satisfied by $y(t) = 2 - 3te^{3t}$.

(Do not find the initial values which make this the solution.)

$r(r-3)^2 = 0 \Rightarrow r(r^2 - 6r + 9) = r^3 - 6r^2 + 9r = 0$

$y^{(3)} - 6y^{(2)} + 9y^{(1)} = 0.$

5. *(2+8+8pts)* Given the differential equation $x^2y'' - x(x+2)y' + (x+2)y = 0$ (for $x > 0$).

(A) Verify that $y_1(x) = xe^x$ is a solution.

(B) Use the method of **reduction of order** to find a second solution $y_2(x)$.
Verify that your answer is a solution.

$(xe^x)' = (x+1)e^x$
$[(x+1)e^x]' = (x+2)e^x$

$y = uxe^x$
$y' = u'xe^x + u(x+1)e^x$
$y'' = u''xe^x + 2u'(x+1)e^x + u(x+2)e^x$

$x^2y'' - x(x+2)y' + (x+2)y = 0$

$x^2[u''xe^x + 2(x+1)e^x u' + (x+2)e^x u] - x(x+2)\{xe^x u' + (x+1)e^x u\}$

$+ (x+2)xe^x u = 0.$

$x^3 + 2x^2$ $\quad -x^3 - 3x^2 - 2x + x^2 + 2x$

Collect terms:

$x^3e^xu'' + u'(2x^3 + 2x^2 - x^3 - 2x^2)e^x u' + u(x^2(x+2) - x(x+2)(x+1) + x(x+2))$
$e^x u = 0$

$\Rightarrow x^3e^xu'' + x^3e^xu' = 0$

$\Rightarrow u'' + u' = 0$ say $z = u'$ $\Rightarrow z' + z = 0$, $z' = -z$ $\Rightarrow z = e^{-x}$.

$\Rightarrow u' = z = e^{-x}$ $\Rightarrow u = -e^{-x} \Rightarrow y_2 = x(e^x)(-e^{-x}) = -x$ OR $y_2 = x$.

(C) Find the general solution to the **nonhomogeneous** differential equation

$$x^2y'' - x(x+2)y' + (x+2)y = 2x^3 \quad (\text{for } x > 0).$$

One can observe $y = -2x^2$ solves the above equation.
Otherwise apply variation of parameters.

$y = xu_1 + xe^xu_2$ assumption: $\boxed{\begin{array}{l} u_1'x + u_2'xe^x = 0 \\ u_1' + u_2'e^x = 0 \end{array}}$

$y' = u_1 + (x+1)e^x u_2$

$y'' = u_1' + (x+1)e^x u_2' + (x+2)e^x$

ANSWER:
$\boxed{y = -2x^2 + c_1x + c_2xe^x}$

$x^2y'' - x(x+2)y' + (x+2)y = 2x^3$

$2x^3 = x^2[u_1' + (x+1)e^xu_2' + (x+2)e^x] - x(x+2)[u_1 + (x+1)e^xu_2] - (x+2)[xu_1 + xe^xu_2]$

$2x^3 = x^2u_1' + (x+1)x^2e^xu_2'$

$\boxed{\begin{array}{l} y = -2x \cdot x + -2e^{-x} \cdot xe^x \\ y = -2x^2 - 2x \end{array}}$

$\Rightarrow 2x = u_1' + (x+1)e^xu_2'$
$0 = u_1' + e^xu_2'$
(ASSUMPTION)

$2x = xe^xu_2'$
$u_2' = 2e^{-x}$ $\Rightarrow u_2 = -2e^{-x}$
$u_1' = -e^x \cdot u_2' = -2 \Rightarrow u_1 = -2x.$

6. *(12+4pts)* The following parts are about the method of undetermined coefficients.

(A) Solve $y'' - 2y' - 3y = 5e^{2t} - 3te^{2t}$ with $y(0) = 1$ and $y'(0) = -1$.

(Use the method of **undetermined coefficients.**)

(i) Homogeneous part:

$y'' - 2y' - 3y = 0.$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = -1,\ r = 3$

Fund. sol'ns: e^{-t}, e^{3t}

(ii) Undet coeffs:

$y = y_p = (At+B)e^{2t}$

$y' = (At+B)2e^{2t} + Ae^{2t}$

$y' = (2At+2B+A)e^{2t}$

$y'' = 2\{2At+2B+A\}e^{2t} + 2Ae^{2t}$

$= \{4At+4B+4A\}e^{2t}$

$y'' - 2y' - 3y = (5-3t)e^{2t}$

$4At + 4B + 4A - 2(2At + A + 2B) - 3(At + B) = 5 - 3t$

$(4A - 4A - 3A)t + (4A - 2A + 4B - 4B - 3B) = -3t + 5$

$-3A\ t + (2A - 3B) = -3t + 5 \Rightarrow \boxed{A = 1 \quad B = -1}$

$\boxed{y = (t-1)e^{2t}}$

IVP:

$y = (t-1)e^{2t} + c_1 e^{-t} + c_2 e^{3t}$

$1 = y(0) = (0-1)\cdot 1 + c_1 + c_2$

$\Rightarrow \boxed{c_1 + c_2 = +2}$

$y' = 2(t-1)e^{2t} + e^{2t} - c_1 e^{-t} + 3c_2 e^{3t}$

$-1 = y'(0) = 2(-1)\cdot 1 + 1 - c_1 + 3c_2$

$-1 + 2 - 1 = 0$

$\boxed{-c_1 + 3c_2 = 0}$

$c_1 + c_2 = +2$

$-c_1 + 3c_2 = 0$

$4c_2 = +2$

$c_2 = +\frac{1}{2} \quad c_1 = +\frac{3}{2}$

$$\boxed{y = (t-1)e^{2t} + \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t}}$$

(B) Give the **form** for the particular solution Y used in the method of undetermined coefficients to solve

$$y^{(5)} - 5y^{(4)} + 12y''' - 16y'' + 12y' - 4y = e^t - t\sin(t) + te^t\cos(t)$$

if $y^{(5)} - 5y^{(4)} + 12y''' - 16y'' + 12y' - 4y = 0$ has fundamental solutions

$$y_1 = e^t\cos(t), \quad y_2 = e^t\sin(t), \quad y_3 = te^t\cos(t), \quad y_4 = te^t\sin(t), \quad y_5 = e^t.$$

$$Y = Ate^t + t^2(C_1 + C_2 t)e^t\cos t + t^2(D_1 + D_2 t)e^t\sin t$$

$$+ (F_1 + F_2 t)\cos t + (G_1 + G_2 t)\sin t$$